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# The appropriate edge conditions for two-dimensional quasicrystal semi-infinite strips with mixed edge-data

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ABSTRACT

For two-dimensional quasicrystal semi-infinite strips with mixed edge-data, the Betti–Rayleigh reciprocal theorem and the general solution of plane elasticity of quasicrystals are applied in a novel way to obtain the appropriate edge conditions accurate to all order. By introducing two definitions for the decaying and regular states, the necessary conditions deduced form the reciprocal theorem, for the edge-data to induce only a decaying elastostatic are directly translated into the appropriate edge conditions for the existence of a rapidly decaying solution of the strips. Once a suitable regular state, which fulfills load-free boundary conditions, is constructed for the relevant edge-data, the translation is immediate. However, this is not the situation for general edge-data. For the case of transverse bending and in-plane extension of the strips, these decaying state conditions are obtained explicitly for the first time when the mixed edge-data are imposed on the strip edge. Besides, for a degenerated form, an analytical solution of the decaying state is formulated to verify validity of these edge conditions.


## 1. Introduction

Quasicrystals (QCs) (solids with a long-range orientational order and a long-range quasiperiodic translational order (Levine and Steinhardt, 1984)) as a new structure of solid matter were discovered around 1984 (Shechtman et al., 1984). The discovery has brought a significant breakthrough for condensed matter physics in recent years. The electronic structure and the optic, magnetic, thermal and mechanical properties of the material have been extensively investigated in experimental and theoretical analyses (Socolar et al., 1986; Ronchetti, 1987; Ovid'ko, 1992; Wollgarten et al., 1993), which show their complex structure and unusual properties. Elasticity is one of the interesting properties of QCs. Within the framework of the Landau–Lifshitz phenomenological theory, the elastic energies of QCs were formulated (Bak, 1985; Levine and Steinhardt, 1986). In particular, the field of linear elasticity theory of QCs has been investigated for many years (Ding et al., 1993; Yang et al., 1993; Hu et al., 1996; Wang et al., 1997). It provides us with a fundamental theory based on the notion of a continuum model to describe the elastic behavior of QCs. For a comprehensive review in this field, the readers are referred to the works by Hu et al. (2000) and Fan and Mai (2004).

Under external loads, the exact solution of linear elastostatic problems for slender and thin bodies consists of an interior component significant throughout the bodies and an outer (boundary layer) component in a decaying form. Near a lateral edge, the interior component is supplemented by boundary layer component which becomes insignificant away from the edge. The prescribed admissible boundary conditions can be satisfied only by a combination of these two components. However, the boundary layer solution, even just a leading term approximation, needed to fit the edge-data is rather intractable except for cases with simple geometries and load symmetries. This and the fact that the solution behavior near the edges is often not needed from practical viewpoint have driven people to take efforts over the years to formulate the interior solution, by assigning an appropriate portion of the prescribed edge-data to it, without any reference to the boundary layer solution. Gregory and Wan (1984, 1985, 1988) and Wan (2003) developed a novel method determining the interior solution successfully and effectively, and provided the results for several plate problems. Through generalizing the method, a set of necessary conditions on the edge-data for the existence of a rapidly decaying solution is established, and various extensions have been found among one-dimensional (1D) hexagonal QC plates (Gao et al., 2007a) and two-dimensional (2D) dodecagonal QC plates (Gao et al., 2007b).

Slender and thin bodies are one of the most well known structures of vital significance in the structural design and therefore received extensive study from scientific workers. To set up the general framework of the QC strip theories, the appropriate stress and mixed edge conditions accurate to all order for the strips of general edge geometry and loadings should be formulated explicitly. These provide important information for studying the defor-

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mation and mechanical/physical behavior of the new solid phase and understanding clearly the interplay of the interaction between the phonon and phason activity. They also play an important role in numerical simulations such as the finite element method and the boundary element method.

It should be emphasized that the previous studies were undertaken within the scope of elastic or QC plates. Otherwise, to the author's knowledge, relevant edge conditions for 2D QC semi-infinite strips are still very rare in the literature. Due to its importance, a parallel development of edge conditions for 2D QC semi-infinite strips should be useful as well. In the following sections, we obtain a set of necessary conditions on the edge-data for the existence of a rapidly decaying solution. By generalizing the model and method for plates (Gao et al., 2007a,b) to semi-infinite strips and by invoking the general solution of 2D QCs, these necessary conditions are then translated into the desired set of edge conditions for transverse bending and in-plane extension.

## 2. Basic equations and the general solution

For a 2D QCs referred to a Cartesian coordinate system $(x, y, z)$, let $x$–$y$ plane be the quasi-periodic plane and $z$ be the periodic direction. Consider a homogeneous linear QC semi-infinite strip that occupies the region: $x \geqslant 0$, $-1/2 \leqslant y \leqslant 1/2$, $-h \leqslant z \leqslant h$, where $2h$ is the height of the body. The strip is in an equilibrium state of plane strain under no external load in the interior and no tractions on the top and bottom surfaces.

2D decagonal and octagonal QCs are divided into four Laue classes, but only Laue classes 14 and 16 possess three orthogonal symmetry planes. The point groups $10mm$, 1022, $10\bar{m}2$ and $10/mmm$ belong to Laue class 14, and the point groups $8mm$, 822, $8\bar{m}2$ and $8/mmm$ belong to Laue class 16. Following Hu et al. (1996), it is clear that there is a similarity between the general equations of the two Laue classes. In the absence of body forces, the general equations governing the plane strain state of the two Laue classes can be written as:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\partial_\beta u_\alpha + \partial_\alpha u_\beta), \quad w_{x\beta} = \partial_\beta w_x, \tag{1}$$

$$\partial_\beta \sigma_{\alpha\beta} = 0, \quad \partial_\beta H_{x\beta} = 0, \tag{2}$$

$$\begin{aligned} &\sigma_{xx} = C_{11}\varepsilon_{xx} + C_{13}\varepsilon_{zz} + Rw_{xx},\\ &\sigma_{zz} = C_{13}\varepsilon_{xx} + C_{33}\varepsilon_{zz},\\ &\sigma_{zx} = \sigma_{xz} = 2C_{44}\varepsilon_{zx},\\ &H_{xx} = R\varepsilon_{xx} + K_1 w_{xx},\\ &H_{xz} = K_4 w_{xz}, \end{aligned} \tag{3}$$

where $\alpha$, $\beta = x, z$, $u_\alpha$ and $w_x$ denote phonon and phason displacements in the physical and perpendicular spaces, respectively, $\sigma_{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ phonon stresses and strains, respectively, $H_{x\beta}$ and $w_{x\beta}$ phason stresses and strains, respectively, $C_{11}$, $C_{13}$, $C_{33}$, $C_{44}$, $K_1$ and $K_4$ elastic constants in the phonon and phason fields, respectively, $R$ phonon–phason coupling elastic constant.

According to the general solution of plane elasticity of 2D QCs (Gao et al., 2008a), the components of displacements take the form:

$$u_x = \delta_{Ii}\partial_x\psi_i, \quad u_z = m_i\partial_z\psi_i, \quad w_x = l_i\partial_x\psi_i, \tag{4}$$

where $i = 1, 2, 3$, $\delta_{ij}$ is the Kronecker delta symbol, and the following summation convention has been used throughout this paper: the Einstein summation over repeated lower case indices from 1 to 3 is applied, while upper case indices take on the same numbers as the corresponding lower case ones but are not summed. Besides, the potential functions $\psi_i$ satisfy the equations

$$\nabla_I^2\psi_i = \partial_x^2\psi_i + \frac{1}{s_I^2}\partial_z^2\psi_i = 0. \tag{5}$$

The values of $m_i$, $l_i$ and $s_i^2$ are related by the following expressions:

$$\frac{C_{33}m_i}{C_{13} + C_{44}(1 + m_i)} = \frac{C_{13}m_i + C_{44}(1 + m_i)}{C_{11} + Rl_i} = \frac{K_4 l_i}{R + K_1 l_i} = \frac{1}{s_i^2}, \tag{6}$$

where $s_i^2$ are three characteristic roots (or eigenvalues) of the following cubic algebra equation of $s^2$

$$as^6 - bs^4 + cs^2 - d = 0. \tag{7}$$

The constants in the preceding equations are

$$\begin{aligned} &a = C_{33}C_{44}K_4,\\ &b = C_{33}C_{44}K_1 + (C_{11}C_{33} - 2C_{13}C_{44} - C_{13}^2)K_4,\\ &c = C_{11}C_{44}K_4 + (C_{11}C_{33} - 2C_{13}C_{44} - C_{13}^2)K_1 - C_{33}R^2,\\ &d = C_{11}C_{44}K_1 - C_{44}R^2. \end{aligned} \tag{8}$$

For the sake of brevity and conciseness, the appropriate edge conditions for semi-infinite strips will be given only to the case of distinct eigenvalues $s_i^2$ in the following context. When equal eigenvalues appear, the edge conditions for these cases can be obtained by using a similar analysis technique, although the general solution will take a more complicated form for these cases (Gao et al., 2008a,b).

## 3. Necessary conditions for a decaying state

For a semi-infinite strip, all surfaces of the body will be traction free except for the surface $x = 0$, where a prescribed self-equilibrated traction is applied. For this type of loading, it is assumed that the stresses decay to zero as $x \to \infty$. The top and bottom faces of the strips are taken to be traction free, so that

$$\sigma_{xz} = \sigma_{zz} = 0, \quad H_{xz} = 0 (z = \pm h). \tag{9}$$

The presence of any body or surface loads may be removed by a particular solution. Then the only forcing terms in the problem are prescribed on the end $x = 0$ in terms of stress or displacement edge-data in the form of one of the following eight admissible combinations,

Case A:

$$\begin{aligned} &\sigma_{xx}(0,z) = \bar{\sigma}_{xx}(z), \quad \sigma_{xz}(0,z) = \bar{\sigma}_{xz}(z), \quad H_{xx}(0,z) = \overline{H}_{xx}(z),\\ &H_{xz}(0,z) = \overline{H}_{xz}(z), \end{aligned} \tag{10}$$

Case B:

$$\begin{aligned} &u_x(0,z) = \bar{u}_x(z), \quad \sigma_{xz}(0,z) = \bar{\sigma}_{xz}(z), \quad w_x(0,z) = \bar{w}_x(z),\\ &H_{xz}(0,z) = \overline{H}_{xz}(z), \end{aligned} \tag{11}$$

Case C:

$$\begin{aligned} &\sigma_{xx}(0,z) = \bar{\sigma}_{xx}(z), \quad u_z(0,z) = \bar{u}_z(z), \quad H_{xx}(0,z) = \overline{H}_{xx}(z),\\ &H_{xz}(0,z) = \overline{H}_{xz}(z), \end{aligned} \tag{12}$$

Case D:

$$\begin{aligned} &u_x(0,z) = \bar{u}_x(z), \quad u_z(0,z) = \bar{u}_z(z), \quad w_x(0,z) = \bar{w}_x(z),\\ &H_{xz}(0,z) = \overline{H}_{xz}(z), \end{aligned} \tag{13}$$

Case E:

$$\begin{aligned} &\sigma_{xx}(0,z) = \bar{\sigma}_{xx}(z), \quad \sigma_{xz}(0,z) = \bar{\sigma}_{xz}(z), \quad w_x(0,z) = \bar{w}_x(z),\\ &H_{xz}(0,z) = \overline{H}_{xz}(z), \end{aligned} \tag{14}$$

Case F:

$$\begin{aligned} &u_x(0,z) = \bar{u}_x(z), \quad \sigma_{xz}(0,z) = \bar{\sigma}_{xz}(z), \quad H_{xx}(0,z) = \overline{H}_{xx}(z),\\ &H_{xz}(0,z) = \overline{H}_{xz}(z), \end{aligned} \tag{15}$$

Case G:

$$\sigma_{xx}(0,z)=\bar{\sigma}_{xx}(z),\quad u_z(0,z)=\bar{u}_z(z),\quad w_x(0,z)=\bar{w}_x(z),\quad H_{xz}(0,z)=\overline{H}_{xz}(z), \tag{16}$$

Case H:

$$u_x(0,z)=\bar{u}_x(z),\quad u_z(0,z)=\bar{u}_z(z),\quad H_{xx}(0,z)=\overline{H}_{xx}(z),\quad H_{xz}(0,z)=\overline{H}_{xz}(z). \tag{17}$$

In generalization of analogous statements for QC plates (Gao et al., 2007a,b), two classes of exact states are investigated for the equations of plane strain governing semi-infinite strips with free faces. One of these is designated as the interior state significant throughout the strips. The other complementary class corresponds to a boundary layer solution and is designated as the decaying state. An elastostatic state in the strips is said to be a *regular state*

$$\{u_\alpha,\sigma_{\alpha\beta},w_x,H_{x\beta}\}=O(M_1h^{\lambda})\quad \text{as } h\to 0, \tag{18}$$

or a *decaying state*

$$\{u_\alpha,\sigma_{\alpha\beta},w_x,H_{x\beta}\}=O(M_2e^{-\gamma d/h})\quad \text{as } h\to 0, \tag{19}$$

where $M_1$ and $M_2$ are the maximum modulus for the regular state and decaying state, respectively, $d$ is the minimum distance of the observation point from the edge of the strips, and $M_1$, $M_2$, $\lambda$ and $\gamma$ are positive constants.

Supposing that the edge-data do give rise to the decaying state in the strips, we now apply the Betti–Rayleigh reciprocal theorem for QC media, which takes the form

$$\oiint_S(\sigma^{(1)}_{\alpha\beta}u^{(2)}_\beta+H^{(1)}_{x\alpha}w^{(2)}_x-\sigma^{(2)}_{\alpha\beta}u^{(1)}_\beta-H^{(2)}_{x\alpha}w^{(1)}_x)n_\alpha \mathrm{d}S=0, \tag{20}$$

where $S$ is the surface of the strips which consists of two end planes and a lateral surface, $n_\alpha$ is the direction cosine of the outward normal to $S$. With the foregoing two definitions of elastostatic states in mind, now we take the state with a superscript "(1)" to be the exact solution of the strips, and the decaying state induced by the prescribed edge-data $\bar{\sigma}_{xx}$, $\bar{\sigma}_{xz}$, $\overline{H}_{xx}$, $\bar{u}_x$, $\bar{u}_z$ and $\bar{w}_x$. For the auxiliary state, denoted by superscript "(2)", we take any regular state which fulfills load-free conditions on $S$. Similar to the derivation of necessary conditions for a decaying state in QC plates (Gao et al., 2007a,b), generalizing Gregory and Wan's decay analysis technique to the QC strips, we finally obtain the necessary conditions for a decaying state on the end $x = 0$,

Case A:

$$\int_{-h}^{h}(\bar{\sigma}_{xx}u^{(2)}_x+\bar{\sigma}_{xz}u^{(2)}_z+\overline{H}_{xx}w^{(2)}_x)\mathrm{d}z=0, \tag{21}$$

Case B:

$$\int_{-h}^{h}\left(\bar{u}_x\sigma^{(2)}_{xx}-\bar{\sigma}_{xz}u^{(2)}_z+\bar{w}_xH^{(2)}_{xx}\right)\mathrm{d}z=0, \tag{22}$$

Case C:

$$\int_{-h}^{h}(\bar{\sigma}_{xx}u^{(2)}_x-\bar{u}_z\sigma^{(2)}_{xz}+\overline{H}_{xx}w^{(2)}_x)\mathrm{d}z=0, \tag{23}$$

Case D:

$$\int_{-h}^{h}\left(\bar{u}_x\sigma^{(2)}_{xx}+\bar{u}_z\sigma^{(2)}_{xz}+\bar{w}_xH^{(2)}_{xx}\right)\mathrm{d}z=0, \tag{24}$$

Case E:

$$\int_{-h}^{h}\left(\bar{\sigma}_{xx}u^{(2)}_x+\bar{\sigma}_{xz}u^{(2)}_z-\bar{w}_xH^{(2)}_{xx}\right)\mathrm{d}z=0, \tag{25}$$

Case F:

$$\int_{-h}^{h}(\bar{u}_x\sigma^{(2)}_{xx}-\bar{\sigma}_{xz}u^{(2)}_z-\overline{H}_{xx}w^{(2)}_x)\mathrm{d}z=0, \tag{26}$$

Case G:

$$\int_{-h}^{h}\left(\bar{\sigma}_{xx}u^{(2)}_x-\bar{u}_z\sigma^{(2)}_{xz}-\bar{w}_xH^{(2)}_{xx}\right)\mathrm{d}z=0, \tag{27}$$

Case H:

$$\int_{-h}^{h}(\bar{u}_x\sigma^{(2)}_{xx}+\bar{u}_z\sigma^{(2)}_{xz}-\overline{H}_{xx}w^{(2)}_x)\mathrm{d}z=0. \tag{28}$$

Based on the loadings subjected on the top and bottom surfaces of the strips, the general deformation of the strips can be decomposed into two independent parts: the asymmetric part (transverse bending) and the symmetric part (in-plane extension). In the following two sections, these necessary conditions for the edge-data to induce only a decaying elastostatic state will be translated separately into the appropriate edge conditions for the two deformation forms.

## 4. The appropriate edge conditions for transverse bending

### 4.1. The auxiliary regular states

The main difficulty in performing the preceding process lies in obtaining suitable regular states which satisfy the appropriate boundary conditions. Once a suitable regular state is constructed for the relevant edge-data, the translation is immediate. However, this is not the situation for general edge-data. Now our main task lies in obtaining accurate solutions for these regular states.

We can take a rigid body translation in the $z$-direction as the state 1, i.e.

$$u^{(2)}_x=0,\quad u^{(2)}_z=C,\quad w^{(2)}_x=0,\quad \sigma^{(2)}_{\alpha\beta}=H^{(2)}_{x\beta}=0, \tag{29}$$

where $C$ is a constant.

The state 2 may be taken as a rigid body rotation,

$$u^{(2)}_x=Cz,\quad u^{(2)}_z=-Cx,\quad w^{(2)}_x=0,\quad \sigma^{(2)}_{\alpha\beta}=H^{(2)}_{x\beta}=0. \tag{30}$$

Now, we look for the state 3 with the use of the general solution (4). The potential functions $\psi_i$ are taken as

$$\psi_i=A_ixz, \tag{31}$$

where $A_i$ are unknown constants to be determined later. After taking account of the expressions (6), the displacements and stresses obtained from Eqs. (3), (4) and (31) can be shown to be

$$\begin{aligned}&u^{(2)}_x=\delta_{li}A_iz,\quad u^{(2)}_z=m_iA_ix,\quad w^{(2)}_x=l_iA_iz,\\&\sigma^{(2)}_{xx}=\sigma^{(2)}_{zz}=H^{(2)}_{xx}=0,\quad \sigma^{(2)}_{xz}=\sigma^{(2)}_{zx}=\alpha_iA_i,\quad H^{(2)}_{xz}=\beta_iA_i,\end{aligned} \tag{32}$$

where

$$\alpha_i=C_{44}+C_{44}m_i,\quad \beta_i=K_4l_i. \tag{33}$$

To obtain the state 4, according to the characteristics of transverse bending, by using Eq. (5) we assume

$$\psi_i=B_i(s_I^2z^3-3x^2z), \tag{34}$$

where $B_i$ are unknown constants to be determined later. In virtue of the expressions (6), the displacements and stresses in Eqs. (3), (4) and (34) can be written as

$$\begin{aligned}&u^{(2)}_x=-6\delta_{li}B_ixz,\quad u^{(2)}_z=3m_iB_i(s_I^2z^2-x^2),\quad w^{(2)}_x=-6l_iB_ixz,\\&\sigma^{(2)}_{xx}=-6\alpha_is_I^2B_iz,\quad \sigma^{(2)}_{zz}=6\alpha_iB_iz,\quad \sigma^{(2)}_{xz}=\sigma^{(2)}_{zx}=-6\alpha_iB_ix,\\&H^{(2)}_{xx}=-6\beta_is_I^2B_iz,\quad H^{(2)}_{xz}=-6\beta_iB_ix,\end{aligned} \tag{35}$$

To obtain the state 5, we take the potential functions $\psi_i$ to be of the form

$$\psi_i = C_i(s_l^2 xz^3 - x^3 z) + D_i xz, \tag{36}$$

where $C_i$ and $D_i$ are unknown constants yet to be determined. In terms of the expressions (6), the displacements and stresses in Eqs. (3), (4) and (36) have the following form as

$$\begin{aligned}
u_x^{(2)} &= s_i^2 C_i z^3 - 3\delta_{li} C_i x^2 z + \delta_{li} D_i z,\\
u_z^{(2)} &= m_i[C_i(3s_l^2 xz^2 - x^3) + D_i x],\\
w_x^{(2)} &= l_i s_l^2 C_i z^3 - 3 l_i C_i x^2 z + l_i D_i z,\\
\sigma_{xx}^{(2)} &= -6\alpha_i s_l^2 C_i xz, \quad \sigma_{zz}^{(2)} = 6\alpha_i C_i xz,\\
\sigma_{xz}^{(2)} &= \sigma_{zx}^{(2)} = 3\alpha_i C_i (s_l^2 z^2 - x^2) + \alpha_i D_i,\\
H_{xx}^{(2)} &= -6\beta_i s_l^2 C_i xz,\\
H_{xz}^{(2)} &= 3\beta_i C_i (s_l^2 z^2 - x^2) + \beta_i D_i.
\end{aligned} \tag{37}$$

### 4.2. The appropriate edge conditions with mixed edge-data

For the case of transverse bending, the appropriate edge conditions can be explicitly determined as follows, at least for the edge-data in Cases A–C and E–G.

#### 4.2.1. Case A

As the procedure in the preceding section indicates, any candidate for regular states must meet load-free conditions (9) and the requirements stipulated below

$$\sigma_{xz} = 0, \quad \sigma_{xx} = 0, \quad H_{xx} = 0 (x = 0). \tag{38}$$

Obviously, the state 1 satisfies the conditions (9) and (38), so the corresponding necessary condition are obtained from Eq. (21)

$$\int_{-h}^{h} \bar{\sigma}_{xz} dz = 0. \tag{39}$$

The second auxiliary regular state may be takes as the state 2, then the corresponding necessary condition is

$$\int_{-h}^{h} \bar{\sigma}_{xx} z dz = 0. \tag{40}$$

Selecting one state from the state 3 or 5 as the third auxiliary regular state, we obtain the third necessary condition for a decaying state when $\overline{H}_{xx}$ is prescribed

$$\int_{-h}^{h} \overline{H}_{xx} z dz = 0. \tag{41}$$

The necessary conditions (39)–(41) are conventional forms of strip theories, although they are formulated explicitly by an application of the reciprocal theorem and the general solution of 2D QCs.

#### 4.2.2. Case B

Regular states must meet the conditions (9) and the requirements

$$\sigma_{xz} = 0, \quad u_x = 0, \quad w_x = 0 (x = 0). \tag{42}$$

As in Case A, selecting a rigid body translations in the state 1 as the first auxiliary regular state, we certainly must have the corresponding necessary conditions (39).

We take the state 4 as the second auxiliary regular state. On substituting Eq. (35) into Eqs. (9) and (42), we can determine the relationship among these unknown constants as

$$\frac{B_1}{\lambda_1} = \frac{B_2}{\lambda_2} = \frac{B_3}{\lambda_3}, \quad \lambda_i = e_{ijk}\alpha_j\beta_k, \tag{43}$$

where $e_{ijk}$ is Levi-Civita permutation symbol with indices varying from 1 to 3. Inserting this auxiliary regular state (35) into Eq. (22), after taking account of the relationship (43), we obtain the second necessary condition for a decaying state when $\bar{u}_x$, $\bar{\sigma}_{xz}$ and $\bar{w}_x$ are prescribed

$$\int_{-h}^{h} \left( \bar{u}_x z + \frac{m_i s_l^2 \lambda_i}{2\alpha_i s_l^2 \lambda_i} \bar{\sigma}_{xz} z^2 + \frac{\beta_i s_l^2 \lambda_i}{\alpha_i s_l^2 \lambda_i} \bar{w}_x z \right) dz = 0. \tag{44}$$

#### 4.2.3. Case C

Consider the conditions on the end $x = 0$,

$$\sigma_{xx} = 0, \quad u_z = 0, \quad H_{xx} = 0 (x = 0). \tag{45}$$

In this case, the states 2 and 3 are chosen as the auxiliary regular states, then the first two necessary conditions are the conditions (40) and (41).

The third auxiliary regular state may be taken as the state 5. Substituting Eq. (37) into Eqs. (9) and (45), we have the relationship among these unknown constants

$$\frac{C_1}{\lambda_1} = \frac{C_2}{\lambda_2} = \frac{C_3}{\lambda_3}, \quad \alpha_i D_i = -3\alpha_i s_l^2 C_i h^2, \quad \beta_i D_i = -3\beta_i s_l^2 C_i h^2. \tag{46}$$

On substituting Eq. (37) into Eq. (23), by using Eqs. (40), (41) and (46) we obtain,

$$\int_{-h}^{h} \left[ \bar{u}_z (h^2 - z^2) + \frac{s_i^2 \lambda_i}{3\alpha_i s_l^2 \lambda_i} \bar{\sigma}_{xx} z^3 + \frac{l_i s_l^2 \lambda_i}{3\alpha_i s_l^2 \lambda_i} \overline{H}_{xx} z^3 \right] dz = 0. \tag{47}$$

#### 4.2.4. Case E

Regular states must meet the requirements on $x = 0$,

$$\sigma_{xz} = 0, \quad \sigma_{xx} = 0, \quad w_x = 0 (x = 0). \tag{48}$$

If the states 1 and 2 are chosen as the auxiliary regular states, then the necessary conditions have the form as the conditions (39) and (40), respectively.

#### 4.2.5. Case F

The conditions on $x = 0$ give

$$\sigma_{xz} = 0, \quad u_x = 0, \quad H_{xx} = 0 (x = 0). \tag{49}$$

If the state 1 is chosen as the auxiliary regular state, then the necessary condition takes the form as the condition (39).

#### 4.2.6. Case G

By noting that

$$\sigma_{xx} = 0, \quad u_z = 0, \quad w_x = 0 (x = 0). \tag{50}$$

Once one state is chosen from the state 2, the necessary condition is the condition (40).

## 5. The appropriate edge conditions for in-plane extension

### 5.1. The auxiliary regular states

Analogous to the deformation of transverse bending, two rigid body translations in the $x$-direction can be chosen as the states 1 and 2

$$u_x^{(2)} = C, \quad u_z^{(2)} = 0, \quad w_x^{(2)} = 0, \quad \sigma_{\alpha\beta}^{(2)} = H_{x\beta}^{(2)} = 0, \tag{51}$$

$$u_x^{(2)} = 0, \quad u_z^{(2)} = 0, \quad w_x^{(2)} = C, \quad \sigma_{\alpha\beta}^{(2)} = H_{x\beta}^{(2)} = 0. \tag{52}$$

The state 3 may be taken as the following form,

$$\begin{aligned}
&u_x^{(2)} = Cx, \quad u_z^{(2)} = -\tfrac{C_{13}}{C_{33}} Cz, \quad w_x^{(2)} = 0,\\
&\sigma_{xx}^{(2)} = \tfrac{C_{11}C_{33} - C_{13}^2}{C_{33}} C, \quad H_{xx}^{(2)} = RC, \quad \sigma_{zz} = \sigma_{xz} = H_{xz} = 0.
\end{aligned} \tag{53}$$

To obtain the state 4, with the features of in-plane extension we take the potential functions $\psi_i$ to be of the form

$$\psi_i = E_i(s_I^2 z^2 - x^2), \tag{54}$$

where $E_i$ are unknown constants to be determined later. The displacements and stresses obtained from Eqs. (3), (4) and (54) are expressed as

$$\begin{aligned} &u_x^{(2)} = -2\delta_{Ii}E_i x, \quad u_z^{(2)} = 2m_i s_I^2 E_i z, \quad w_z^{(2)} = -2l_i E_i x, \\ &\sigma_{xx}^{(2)} = -2\alpha_i s_I^2 E_i, \quad \sigma_{zz}^{(2)} = 2\alpha_i E_i, \quad \sigma_{xz}^{(2)} = \sigma_{zx}^{(2)} = H_{zx}^{(2)} = 0, \\ &H_{xx}^{(2)} = -2\beta_i s_I^2 E_i. \end{aligned} \tag{55}$$

### 5.2. The appropriate edge conditions with mixed edge-data

For the following edge-data, the strips of in-plane extension meet the same load-free conditions (9) and the requirements on the end $x = 0$ as those of transverse bending.

#### 5.2.1. Cases A and C

If the states 1 and 2 are chosen as the auxiliary regular states, then we obtain the same necessary conditions for Cases A and C

$$\int_{-h}^{h} \bar{\sigma}_{xx} \mathrm{d}z = 0, \tag{56}$$

$$\int_{-h}^{h} \bar{H}_{xx} \mathrm{d}z = 0. \tag{57}$$

#### 5.2.2. Case B

In this case, the auxiliary regular state may be taken as the state 3. Inserting Eq. (53) into Eq. (22), we obtain the necessary condition for a decaying state when $\bar{u}_x$, $\bar{\sigma}_{xz}$ and $\bar{w}_x$ are prescribed

$$\int_{-h}^{h} \left( \bar{u}_x + \frac{C_{13}}{C_{11}C_{33} - C_{33}^2} \bar{\sigma}_{xz} z + \frac{RC_{33}}{C_{11}C_{33} - C_{33}^2} \bar{w}_x \right) \mathrm{d}z = 0. \tag{58}$$

#### 5.2.3. Case E

Selecting the state 1 as the first auxiliary regular condition, we certainly must have the necessary condition (56).

We take the state 4 as the second auxiliary regular state. On substituting Eq. (55) into Eqs. (9) and (48), we can determine the relationship among these unknown constants as

$$\frac{E_1}{\xi_1} = \frac{E_2}{\xi_2} = \frac{E_3}{\xi_3}, \quad \xi_i = e_{ijk}\alpha_j \alpha_k s_K^2. \tag{59}$$

With the help of Eqs. (55) and (59), the second necessary condition for a decaying state is obtained from Eq. (25) when $\bar{\sigma}_{xz}$ and $\bar{w}_x$ are prescribed

$$\int_{-h}^{h} \left( \bar{w}_x + \frac{m_i s_I^2 \xi_i}{\beta_i s_I^2 \xi_i} \bar{\sigma}_{xz} z \right) \mathrm{d}z = 0. \tag{60}$$

#### 5.2.4. Case F

If the state 2 is chosen as the auxiliary regular state, then the necessary condition takes the form as the condition (57).

The second auxiliary regular state may be taken as the state 4. Substituting Eq. (55) into Eqs. (9) and (49), we have the relationship among these unknown constants

$$\frac{E_1}{\eta_1} = \frac{E_2}{\eta_2} = \frac{E_3}{\eta_3}, \quad \eta_i = e_{ijk}\alpha_j \beta_k s_K^2. \tag{61}$$

On substituting Eq. (55) into Eq. (26), by using the relationship (61) we obtain,

$$\int_{-h}^{h} \left( \bar{u}_x + \frac{m_i s_I^2 \eta_i}{\alpha_i s_I^2 \eta_i} \bar{\sigma}_{xz} z \right) \mathrm{d}z = 0. \tag{62}$$

#### 5.2.5. Case G

If the state 1 is chosen as the auxiliary regular state, then the necessary condition takes the form as the condition (56).

#### 5.2.6. Case H

The conditions on $x = 0$ are

$$u_x = 0, \quad u_z = 0, \quad H_{xx} = 0 (x = 0). \tag{63}$$

When the state 2 is chosen as the auxiliary regular state, we certainly must have the corresponding necessary condition (57).

Up to here, attempts to derive similar results on edge conditions for Cases D, H of transverse bending and Case D of in-plane extension have not been successful, since we have not found any simple regular states suitable for these cases. This lack of success may be related to the fact that no suitable regular states needed for the application of the reciprocal theorem could be found for these cases, thus it is not likely that the desired results are forthcoming. The same is true for isotropic elastic, transversely isotropic elastic and piezoelectric materials (Gregory and Wan, 1984; Lin and Wan, 1988; Gao et al., 2008a,b), since it is not possible to fit the pure displacement edge-data by a regular state of plane strain with $u_\alpha \to 0$ as $x \to \infty$.

For each type of edge-data of two deformation forms, these aforementioned necessary conditions for a decaying state (boundary layer solution) can then be converted into a set of edge conditions appropriate for the interior solution or its various approximate semi-infinite strip theories, which do not involve the boundary layer solution components. As the preceding discussion in Introduction, the difference between the exact solution and the interior one is a decaying state.

To obtain the appropriate edge conditions for a particular semi-infinite strip theory, we should expand in powers of $h$ all terms in all necessary conditions and retain only a suitable number of terms in each expansion. The above results for transverse bending and in-plane extension illustrate the general method for deriving local necessary conditions for a decaying state and therewith appropriate edge conditions for semi-infinite strip theories.

## 6. The degenerated form of 2D QC strips

Determination of the independent elastic constants $C_{ij}$, $K_i$ and $R_i$ for different kinds of QCs depends on their symmetries with the group representation theory. Details may be found in Hu et al. (1996) and Wang et al. (1997). It is noted that, although $C_{ij}$ in QCs can be measured by some experimental methods, $K_i$ are difficult to measure (Tanaka et al., 1996). Recently, significant progress in this area have been made by Jeong and Steinhardt (1993), who evaluated $K_i$ of decagonal QCs by Monte Carlo simulation, and the values of $K_i$ are of the same order of magnitude as $C_{ij}$ obtained from testing (Chernikov et al., 1998). There are no data for $R_i$, which are less than $K_i$ based on estimation by some experts working in the field of QCs.

In the above sections, necessary conditions for 2D QC strips are studied. However, up to now the relevant data such as $K_i$ and $R_i$ associated with the present paper are still lacking. Alternatively, we will discuss a degenerated form of 2D QC strips to investigate its validity, i.e. 2D QC body reduces to transversely isotropic elastic body.

### 6.1. The appropriate edge conditions for transversely isotropic elastic strips

In this case, no phonon–phason field coupling effect is taken into account, i.e. $R = 0$. Hence the governing Eqs. (1)–(3) reduce to two groups of equations for uncoupled phonon and phason field problems, respectively. Then, the cubic Eq. (7) of $s^2$ can be reformulated as

$$[C_{33}C_{44}s^4 + (C_{13}^2 + 2C_{13}C_{44} - C_{11}C_{33})s^2 + C_{11}C_{44}](K_4s^2 - K_1) = 0. \tag{64}$$

Let $s_1^2$ and $s_2^2$ be the roots of the first multiplier of Eq. (64), i.e.

$$C_{33}C_{44}s^4 + (C_{13}^2 + 2C_{13}C_{44} - C_{11}C_{33})s^2 + C_{11}C_{44} = 0 \tag{65}$$

and $s_3^2 = K_1/K_4$ be the root of the second multiplier of Eq. (64) with no loss of generality. Then it can be seen that $s_1^2$ and $s_2^2$ relate only to elastic constants in the phonon field, while $s_3^2$ associates only with elastic constants in the phason field.

In the transversely isotropic elastic body, the constants $m_i$ and $l_i$ degenerated from expressions (6) have the following form in the use of Eq. (65)

$$m_i = \frac{C_{11} - C_{44}s_i^2}{(C_{13} + C_{44})s_i^2} = \frac{C_{13} + C_{44}}{C_{33}s_i^2 - C_{44}}, \quad l_i = 0, \tag{66}$$

where $i = 1, 2$. On the other hand, $m_3 = 0$ and $l_3 \neq 0$, which associates with $s_3^2$. Since the analysis in the following calculation does not involve $m_3$ and $l_3$ except the requirement $l_3 \neq 0$, it suffices to discuss only $m_i$ and $s_i^2$. The following identities can be proved on the basis of Eqs. (65) and (66)

$$\begin{aligned} &m_1m_2 = 1, \quad m_1 - m_2 = \tfrac{C_{33}}{C_{13}+C_{44}}(s_2^2 - s_1^2), \\ &s_1^2s_2^2 = \tfrac{C_{11}}{C_{33}}, \quad s_1^2 + s_2^2 = -\tfrac{C_{13}^2+2C_{13}C_{44}-C_{11}C_{33}}{C_{33}C_{44}}. \end{aligned} \tag{67}$$

From Eqs. (66) and (67), we have expressions of Eqs. (44), (47) and (58) in the simplified form as

$$\int_{-h}^{h} \left( \bar{u}_x z + \frac{1}{2}\frac{C_{13}}{C_{11}C_{33} - C_{13}^2}\bar{\sigma}_{xz}z^2 \right) dz = 0, \tag{68}$$

$$\int_{-h}^{h} \left[ \bar{u}_z(h^2 - z^2) + \frac{1}{3}\frac{C_{11}C_{33} - C_{13}^2 - C_{13}C_{44}}{C_{44}(C_{11}C_{33} - C_{13}^2)}\bar{\sigma}_{xx}z^3 \right] dz = 0, \tag{69}$$

$$\int_{-h}^{h} \left( \bar{u}_x z + \frac{C_{13}}{C_{11}C_{33} - C_{13}^2}\bar{\sigma}_{xz}z^2 \right) dz = 0. \tag{70}$$

Up to here, Eqs.(39), (40), (68) and (69) together constitute the necessary conditions for transverse bending, while Eqs. (56) and (70) for in-plane extension, in which $\bar{u}_\alpha$ and $\bar{\sigma}_{x\beta}$ must satisfy in order to give rise only to a decaying state within the strips. Noticeably, the above result is the same as the corresponding one deduced by Lin and Wan (1988), although the two approaches are appreciably different. Therefore, the appropriate edge conditions of 2D QC strips can be degenerated into those of transversely isotropic elastic strips by omitting the phonon–phason fields coupling effect. In comparison with the appropriate edge conditions of transversely isotropic elastic strips, the existence of phason field influences strongly the deformation and mechanical behavior of QC materials. A theoretical description of the deformed state of QCs requires a combined consideration of interrelated phonon and phason fields, so the elasticity of QCs is more complex than that of the conventional crystals.

### *6.2. Analytical solution of transversely isotropic elastic strips*

Now we verify whether the necessary conditions Eqs.(39), (40), (56), (68), (69) and (70) are right or not. For stress or mixed data on the end $x = 0$, the relevant necessary conditions for the absence of the interior state in the induced bending or extension deformations can now be deduced from analytical solution of the decaying state. In terms of Airy stress function $\Phi$, the solution of plane problem takes the form

$$\sigma_{xx} = \partial_z^2\Phi, \quad \sigma_{zz} = \partial_x^2\Phi, \quad \sigma_{xz} = -\partial_x\partial_z\Phi, \tag{71}$$

where

$$\nabla_1^2\nabla_2^2\Phi = 0. \tag{72}$$

For transverse bending, it is apparent that $\Phi$ has solution of the form

$$\Phi = P_ih^2 \sin\left(\frac{s_iz}{h}\right)e^{-\frac{x}{h}}(i = 1, 2), \tag{73}$$

where $P_i$ are unknown constants to be determined, and the Einstein summation is applied from 1 to 2 in this subsection. Thus, the stresses and displacements can be shown to be

$$\begin{aligned} &\sigma_{xx} = -P_is_l^2 \sin\left(\tfrac{s_lz}{h}\right)e^{-\frac{x}{h}}, \quad \sigma_{zz} = P_i \sin\left(\tfrac{s_lz}{h}\right)e^{-\frac{x}{h}}, \quad \sigma_{xz} = P_is_l \cos\left(\tfrac{s_lz}{h}\right)e^{-\frac{x}{h}}, \\ &u_x = \tfrac{(C_{13}+C_{44})s_l^2h}{C_{44}(C_{13}s_l^2+C_{11})}P_i \sin\left(\tfrac{s_lz}{h}\right)e^{-\frac{x}{h}}, \quad u_z = \tfrac{(C_{44}s_l^3-C_{11}s_l)h}{C_{44}(C_{13}s_l^2+C_{11})}P_i \cos\left(\tfrac{s_lz}{h}\right)e^{-\frac{x}{h}}. \end{aligned} \tag{74}$$

It can find that the field (74) is exponentially small in the interior of the strips as $h \to 0$. As the definition of the decaying state indicates, the field (74) is a decaying state in the strips.

We can determine $P_i$ by inserting Eq. (74) into the load-free conditions (9) on the top and bottom faces of the strips which yields

$$P_1 = \frac{1}{\sin s_1}, \quad P_2 = -\frac{1}{\sin s_2}, \tag{75}$$

provided that $s_i$ satisfy

$$\frac{s_1 \cos s_1}{\sin s_1} = \frac{s_2 \cos s_2}{\sin s_2}. \tag{76}$$

For the decaying state, we can testify by direct substitution of Eq. (74) that these necessary conditions are indeed satisfied by the corresponding decaying field (74) because of Eqs. (75) and (76). Therefore, the stress and mixed edge-data must satisfy the appropriate edge conditions 39, 40, 68 and 69, if the data give rise only to the decaying state. After the same manipulation as the case of transverse bending, we can make sure that the necessary conditions (56) and (70) are also right for in-plane extension.

Of course the theoretical prediction needs to be verified by experimental observation. In principle, experimental methods can be used to determine the displacement and stress fields of the materials, but there are few such results to date. So there are difficulties in comparing theoretical solutions with test data apart from those obtained in the present paper.

## 7. Conclusions

In this paper we extend the model and method for elastic and QC plates to 2D QC semi-infinite strips in transverse bending and in-plane extension, which enables us to formulate the correct edge conditions of QC semi-infinite strips with the mixed edge-data for the first time. However, attempts to derive the corresponding edge conditions for displacement and other types of edge-data have not been successful. We have not found any simple auxiliary regular states suitable for these edge-data, but this does not mean that our approach is useless in these cases. It means that the required auxiliary states are themselves the solutions of certain particular boundary value problems, which, when solved once and for all, are to be used in the appropriate decaying state conditions.

Due to the complexity of QC elasticity, there is no possibility to find the general and universal forms of the correct edge conditions suitable for all other QC materials. Fortunately, the edge conditions for this QC strip can be exactly reduced to some standard edge conditions, thus providing the opportunity to: (1) determine rapidly decaying states within the strip, (2) formulate the interior solution of the QC strip problems and (3) extend the description of these decaying state conditions for the deformations of transverse bending and in-plane extension approximately to other QC materials, etc.

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